RENORMALISING NN SCATTERING: IS POWER COUNTING POWERLESS?

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The renormalisation of NN scattering in theories with zero-range interactions is examined using a cut-off regularisation where the cut-off is taken to infinity, dimensional regularisation (DR) with minimal subtraction, and DR with power-divergence subtraction. In the infinite cut-off limit power counting breaks down: terms of different orders in the potential contribute to the scattering amplitude at the same order. Minimal subtraction does yield a systematic expansion, but with a very limited range of validity for systems that have unnaturally large scattering lengths. For a finite cut-off, the behaviour of the couplings as the cut-off is lowered shows that a theory with a natural scattering length approaches an IR fixed point. In the corresponding effective theory, loop corrections can be treated perturbatively. In contrast, if there is an IR fixed point for systems with an infinite scattering length it must be a nonperturbative one, with no power counting. For such systems, power-divergence subtraction appears to yield a systematic expansion, but with a different power counting from Weinberg's. However the scheme omits IR divergent terms that would otherwise lead to nonperturbative behaviour and so the interpretation of the fixed point remains unclear.

1   Introduction

The possibility of applying the techniques of of effective field theory (EFT) to nuclear physics was first raised by Weinberg[1] when he wrote down power-counting rules for the low-momentum expansion of the *irreducible* NN scattering amplitude. The raised the possibility of applying the techniques of chiral perturbation theory (ChPT)[2,3,4] to nuclear forces.

By focussing on the NN potential (more precisely the two-nucleon irreducible scattering amplitude) one avoids contributions where the two intermediate nucleons are almost on-shell, and so have small energy denominators of order $\mathcal{O}(p^2/M)$ instead of $\mathcal{O}(p)$. However, this is the physics responsible for nuclear binding, and so to describe nuclei with an EFT it is not enough to write down a potential: one must be able to iterate it, by solving a Schrödinger or Lippmann-Schwinger equation.

At this point one encounters a problem. The EFT is based on a Lagrangian with local meson-nucleon couplings. As a result, multimeson exchange processes lead to a potential that is highly singular at short distances. To renormalise these short-distance (UV) divergences one has to introduce counterterms in the Lagrangian. These take the form of contact interactions such as

$(\overline{\Psi}\Psi)^2$ (at leading order in the momentum expansion) and $(\overline{\Psi}\Psi)(\overline{\Psi}\nabla^2\Psi)+$H.c. (at next-to-leading order). However, such interactions correspond to δ-function potentials, and the resulting scattering equations only make sense after a further regularisation and renormalisation. Quite a few schemes have been explored recently for this [5−23]. The main question addressed by this workshop is: Can this renormalisation be done while maintaining a useful and systematic organising scheme (power counting) for the potential?

Here I am using systematic to imply that the coefficients appearing in the potential should be meaningful beyond the particular calculation of NN scattering at some order in a momentum expansion that has been used to fix them. This means that higher-order terms in the potential should not contribute to the scattering in the same way as lower-order ones [24], and so changes in the coefficients should be small when includes higher-order terms in the EFT. It also means that the values of these coefficients should be applicable to calculations of other processes [25].

In section 2, I introduce the EFT used to describe s-wave NN scattering at very low energies. The use of a cut-off regulator is described in section 3, along with the problems encountered if one tries to take the cut-off to infinity. Section 4 gives more details of the behaviour if the cut-off is left finite. Approaches based on dimensional regularisation are described in sections 5 (the minimal subtraction scheme) and 6 (power divergence subtraction). In section 7 I compare the results obtained in all these approaches. Apart from the extended discussion of finite cut-offs, these sections are essentially the talk presented at the workshop. Finally, in section 8, I examine the the finite cut-off and power divergence subtraction schemes from a renormalisation-group viewpoint. These ideas, which were developed following discussions at the workshop, suggest that the low-energy EFT of systems with natural scattering lengths corresponds to a perturbative IR fixed point. In contrast, at least for cut-off regularisation, the fixed point for systems with infinite scattering lengths seems to be a nonperturbative one, with no power counting.

2   The model

Like most of the other contributors to this session, I consider an EFT where all mesons (including pions) have been integrated out. Although such a theory is only relevant to extremely low-energy NN scattering, it can be used to address questions of principle. The basic potential in this model contains contact interactions only. To second order in the momentum expansion it has the form

$$V(k',k;E) = C_0 + C_2(k^2 + k'^2) + \cdots, \tag{1}$$

where only the terms relevant to s-wave scattering have been included.

In (1) I have allowed for a possible energy dependence of the potential. This is because, as described below, renormalisation of the scattering equation naturally leads to counterterms proportional to powers of the energy. Energy dependence of the potential should not be too surprising since it can arise whenever degrees of freedom are eliminated from a Schrödinger equation [26]. Indeed energy-dependent contact terms are naturally required to renormalise, for example, contributions to the two-pion exchange potential. Such terms have not normally been considered in EFT treatments of NN scattering, where energy dependence is usually eliminated using the equation of motion. Since energy- and momentum-dependent terms in the potential behave differently off-shell, they can give different results when iterated in a Lippmann-Schwinger equation. For example, with a cut-off regulator that is taken to infinity, the bare couplings are renormalised differently [16]. Hence, until a consistent power-counting scheme has been established, energy-dependent terms ought to be included in the potential.

In treating the scattering non-perturbatively, it is convenient to work with the reactance matrix [27], K, rather than the scattering matrix, T. The off-shell K-matrix for s-wave scattering satisfies a Lippmann-Schwinger equation that is very similar to that for T:

$$K(k', k; E) = V(k', k; E) + \frac{M}{2\pi^2}\mathcal{P}\int_0^\infty q^2 dq \, \frac{V(k', q; E)K(q, k; E)}{p^2 - q^2}. \quad (2)$$

In this expression and throughout this paper, $p = \sqrt{ME}$ denotes the on-shell value of the relative momentum. Note that the definition of K here differs from the more standard one [27] by a factor of $-\pi$. The equation is similar to the one for the T-matrix except that the Green's function satisfies standing-wave boundary conditions. This means that the usual $i\epsilon$ prescription for the integral over q is replaced by the principal value (denoted by $\mathcal{P}$). As a result, the K matrix is real below the threshold for meson production.

The inverse of the on-shell K-matrix differs from that of the on-shell T-matrix by a term $iMp/4\pi$, which ensures that T is unitary if K is Hermitian. This allows the effective-range expansion [28,29,30] to be written as an expansion of $1/K$:

$$
\begin{aligned}
\frac{1}{K(p, p; E)} &= \frac{1}{T(p, p; E)} - \frac{iMp}{4\pi} \\
&= -\frac{M}{4\pi}p\cot\delta(p) \\
&= -\frac{M}{4\pi}\left(-\frac{1}{a} + \frac{1}{2}r_e p^2 + \cdots\right),
\end{aligned}
\quad (3)
$$

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where a is the scattering length and r_e is the effective range.

3 Cut-off regularisation

To regulate the divergences associated with the δ-function potential (1), one can smear it out over distances of the order of $1/\Lambda$ by introducing a separable form factor [16]:

$$V(k', k; E) = f(k'/\Lambda)\left[C_0 + C_2\left(k^2 + k'^2\right)\right]f(k/\Lambda), \qquad (4)$$

where the form factor $f(k/\Lambda)$ satisfies $f(0) = 1$ and falls off rapidly for momenta above the cut-off scale Λ.

The resulting potential has a two-term separable form and so the corresponding Lippmann-Schwinger equation can be solved using standard techniques [27]. The off-shell K-matrix obtained in this way is

$$K(k', k; E) = f(k'/\Lambda)f(k/\Lambda) \qquad (5)$$

$$\times \frac{1 + \frac{C_2}{C_0}(k^2 + k'^2) + \frac{C_2^2}{C_0}\left[I_2(E) - (k^2 + k'^2)I_1(E) + k^2 k'^2 I_0(E)\right]}{\frac{1}{C_0} - I_0(E) - 2\frac{C_2}{C_0}I_1(E) - \frac{C_2^2}{C_0}\left[I_2(E)I_0(E) - I_1(E)^2\right]},$$

where the integrals $I_n(E)$ are given by

$$I_n(E) = \frac{M}{2\pi^2}\mathcal{P}\int_0^\infty \frac{q^{2n+2}f^2(q/\Lambda)}{p^2 - q^2}dq, \qquad (6)$$

with $p^2 = ME$ again. The corresponding expression for T has also been obtained by the Maryland group [13,17], who also considered regulating the theory by simply cutting off the momentum integrals.[a] That procedure leads to a similar expression to (5) but without the factors of $f(k/\Lambda)$ outside.

By expanding the integrals (6) in powers of the energy (or p^2), one can extract their divergent parts:

$$\frac{I_n(E)}{M} = -\sum_{m=0}^n A_m \Lambda^{2m+1} p^{2(n-m)} + \frac{F(p/\Lambda)}{\Lambda}p^{2(n+1)}, \qquad (7)$$

where the dimensionless integrals $A_m > 0$ and $F(p/\Lambda)$ are finite as $\Lambda \to \infty$.[b] Note that, as mentioned above, the divergences appear multiplying powers of

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[a] A similar result has also been obtained by Vall et $al.$ [31], in a rather different notation.
[b] The integral A_m corresponds, up to a factor of $-M\Lambda^{2m+1}$, to I_{2m+1} in the notation of Refs. [13,17].

$E \, (= p^2/M)$ and so it is natural to introduce energy-dependent counterterms to cancel them.

Consider first the limit where $\Lambda \to \infty$. In this limit the outside factors of $f(k/\Lambda)$ in K can be replaced by unity and the final term in (7) vanishes. The on-shell K-matrix is then given by

$$K(p, p; E) = \frac{N(p)}{N(p)A_0 M\Lambda + (1 + C_2 A_1 M\Lambda^3)^2}, \tag{8}$$

where the numerator function is

$$N(p) = C_0 - C_2^2 A_2 M\Lambda^5 + p^2 C_2 (2 + C_2 A_1 M\Lambda^3). \tag{9}$$

For this to remain finite as $\Lambda \to \infty$, the coefficients must vanish like

$$C_0 \sim \frac{1}{M\Lambda}, \qquad C_2 \sim \frac{1}{M\Lambda^3}. \tag{10}$$

This scale dependence of C_0 is that same as that found by Weinberg [1] and Adhikari and coworkers [6].

The leading terms in both the numerator and the denominator of (8) cancel, and so a finite result is obtained from subleading pieces of $C_{0,2}$. If one assumes that $C_{0,2}$ depend analytically on the cut-off [16],

$$C_0 = \frac{\alpha_0}{M\Lambda} + \frac{\beta_0}{\Lambda^2}, \qquad C_2 = \frac{\alpha_2}{M\Lambda^3} + \frac{\beta_2}{\Lambda^4}, \tag{11}$$

then for $\Lambda \to \infty$ the off-shell K-Matrix is

$$K(k', k; E) = \frac{1}{M^2} \frac{\alpha_0 - \alpha_2^2 A_2}{\beta_0 A_0 + 2\beta_2 [A_1 + \alpha_2 (A_1^2 - A_0 A_2)]}, \tag{12}$$

where $\alpha_{0,2}$ satisfy the "fine-tuning" condition

$$\alpha_0 A_0 - \alpha_2^2 A_0 A_2 + (1 + \alpha_2 A_1)^2. \tag{13}$$

Although this result gives a finite scattering length, it has no energy or momentum dependence, and so the effective range is zero.

Therefore, under the assumption of analytic dependence on the cut-off, a non-zero effective range can only be obtained as the cut-off is removed if the coefficients in the potential are allowed to depend on energy. Either or both of the subleading coefficients $\beta_{0,2}$ may be given a linear energy dependence to generate a finite scattering length and effective range. Note that an energy-dependent β_0 leads to the energy appearing the potential with a coefficient of

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order Λ^{-2} while the leading coefficient of the momentum-dependence (C_2) is of order Λ^{-3}. This shows that, in the absence of systematic power counting, energy and momentum dependence need not be equivalent.

The Maryland group[13,17] have taken the on-shell K-matrix (8) and treated it somewhat differently, by demanding that it match the observed scattering length and effective range for any Λ. The results can be expanded as a power series in $\Lambda^{-1/2}$, with the same leading terms as above (10). Terms up to order Λ^{-2} beyond the leading order must be kept to obtain a finite scattering length. Although a finite effective range can be obtained without introducing energy dependence into the potential, this effective range cannot be positive [8,9], as required by Wigner's bound on the momentum dependence of phase shifts [32]. This means that energy dependence of the C's is required to obtain a positive effective range. (A positive scattering length can also be obtained by letting the potential become complex [15]. However this will lead to violations of unitarity for any finite Λ.)

In either case (analytic or nonanalytic dependence on Λ) one finds that terms of different orders in the potential are contributing at the same order in the expansion of K. Indeed the bare parameters are not uniquely determined, if one demands matching to scattering observables only as $\Lambda \to \infty$ [16]. Weinberg's power counting has therefore broken down for cut-off regularisation with $\Lambda \to \infty$.

4   Finite cut-off

If taking the cut-off to infinity destroys power counting, one might hope that keeping it finite could avoid the problem [14]. It does so, but only in natural systems, where $a \sim r_e$. In such cases one can choose a cut-off that is well below the scale of the omitted physics, $\Lambda \ll 1/r_e$, without needing any fine tuning to get the scattering length [17].

If the cut-off is taken to be at or below the scale r_e of the omitted physics, one can no longer omit the terms involving inverse powers of Λ in (7). To order p^2 one has to keep the leading term in the expansion of $F(p/\Lambda)$,

$$F(p/\Lambda) = -B_1 + \mathcal{O}(p^2). \tag{14}$$

The K matrix can then be obtained from (8) by the substitution $A_0 \to A_0 + B_1 p^2/\Lambda^2$, where B_1 is another dimensionless integral. The corresponding effective range expansion can be written

$$\frac{1}{K(p,p;E)} = M\Lambda \left[A_0 + \frac{(1+\hat{C}_2 A_1)^2}{\hat{C}_0 - \hat{C}_2^2 A_2} \right] \tag{15}$$

$$+ \frac{Mp^2}{\Lambda} \left[B_1 - \left(\frac{1+\hat{C}_2 A_1}{\hat{C}_0 - \hat{C}_2^2 A_2} \right)^2 (2 + \hat{C}_2 A_1)\hat{C}_2 \right] + \cdots,$$

in terms of the dimensionless couplings

$$\hat{C}_0 = M\Lambda C_0, \qquad \hat{C}_2 = M\Lambda^3 C_2. \tag{16}$$

For $\Lambda \ll 1/a \sim 1/r_e$, the coefficients $C_{0,2}$ can be expanded in powers of Λ, with the leading behaviours [17]

$$C_0 \sim \frac{r_e}{M} + \mathcal{O}(\Lambda^{-1}), \qquad C_2 \sim \frac{r_e^2}{M\Lambda} + \mathcal{O}(\Lambda^0), \tag{17}$$

instead of (10). Note that the higher-order terms in these coefficients are suppressed by positive powers of Λ, in contrast to the expansion for large cut-offs discussed in the previous section. Loop effects are suppressed by powers of $r_e\Lambda$, and a systematic organisation of the calculation is possible [17]. Note that the $1/\Lambda$ behaviour of C_2 is needed to cancel the contribution B_1/Λ to the effective range in (15). In a natural theory such contributions can be cancelled without spoiling the power counting.

The situation is quite different in systems with bound states close to threshold, such as s-wave NN scattering. In these the scattering length is unnaturally large, $a \gg r_e$. If one chooses a cut-off Λ that is much larger than $1/a$, but still well below $1/r_e$, then one has again to be careful to keep the piece of order $1/\Lambda$ in the effective range term of (15). This $1/\Lambda$ piece is much larger than the effective range and so must be cancelled by a similar piece in C_2, which leads to $C_{0,2}$ having the same dominant behaviour as in (10). However these cannot be regarded as the leading terms in power series in either Λ or $1/\Lambda$, since there are corrections involving powers of $1/\Lambda a$ as well as Λr_e. Also, as in the case of a very large cut-off, C_2 contributes to the scattering length at order unity. Hence, in systems with unnaturally large scattering lengths, there is no systematic power counting. Nonetheless this approach may still be useful as a tool for analysing low-energy NN scattering without introducing too many parameters [5,12,19,22].

5 Minimal subtraction

The predictions of a quantum field theory should be independent of the regulator, yet dimensional regularisation (DR) seems to yield quite different results, evading the problems just discussed [7]. In the minimal subtraction scheme, DR detects only logarithmic divergences., which show up as poles at $D = 4$ dimensions. The loop integrals $I_n(E)$ introduced in (6) above contain only power-law divergences, and so minimal subtraction sets them to be identically zero. As a result the K-matrix is given by the first Born approximation,

$$K(k', k; E) = V(k', k; E). \tag{18}$$

The problem with this scheme is that bound states close to threshold lead to the on-shell K-matrix varying rapidly with energy. In such cases, which the scattering length is unnaturally large, and and the low-momentum expansion of K, and hence also that of the potential, is only valid for $p < \sqrt{2/ar_e}$. [7,11] The minimal subtraction scheme, while systematic, is hardly useful in the case of interest, s-wave NN scattering.

6 Power divergence subtraction

Recently Kaplan, Savage and Wise [20,33] (see also Ref. [21]) have suggested an alternative renormalisation scheme that might allow one to do better using DR. When continued to D space-time dimensions, the loop integrals (6) take the form

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\begin{aligned}
I_n(E) &= \frac{M}{(2\pi)^{D-1}} \left(\frac{\mu}{2}\right)^{4-D} \mathcal{P} \int \frac{q^{2n}}{p^2 - q^2} d^{D-1}\mathbf{q} \tag{19} \\
&= -\frac{M}{(2\sqrt{\pi})^{D-1}} \frac{(\mu/2)^{4-D}}{\Gamma\left(\frac{D-1}{2}\right)} \mathcal{P} \int_0^\infty dx \frac{x^{(D+2n-3)/2}}{x - p^2} \\
&= -\frac{Mp^{2n}}{(2\sqrt{\pi})^{D-1}} \frac{(\mu/2)^{4-D}}{\Gamma(\frac{D-1}{2})} \mathrm{Re}\left[(-p^2)^{(D-3)/2}\right] \Gamma\left(\frac{D+2n-1}{2}\right) \Gamma\left(\frac{3-2n-D}{2}\right).
\end{aligned}
$$

The final Γ-function in this expression has a pole at $D = 3$ for any n. This is the signal of a logarithmic divergence in three dimensions, or a linear one in four. The power-divergence subtraction (PDS) scheme [20] keeps this piece, cancelling it against a counterterm with the same pole at $D = 3$ to leave

$$I_n(E) = -A_0 M \mu p^{2n}, \tag{20}$$

where $A_0 = 1/4\pi$.

The resulting on-shell K-matrix is

$$K(p, p; E) = \left[\frac{1}{C_0 + 2p^2 C_2} + A_0 M \mu \right]^{-1}, \tag{21}$$

and the corresponding scattering length is given by

$$\frac{1}{a} = \frac{4\pi}{M} \left(\frac{1}{C_0} + A_0 M \mu \right). \tag{22}$$

This shows that PDS contains the "strength-range" cancellation needed to give a large scattering length without requiring C_0 to be unnaturally small. As a result one can choose $\mu \gg 1/a$. In this scheme, the scale dependences of $C_{0,2}$ are

$$C_0 \sim \frac{1}{M\mu}, \qquad C_2 \sim \frac{r_e}{M\mu^2}. \tag{23}$$

If one chooses the scale μ to be of the same order as the momenta of interest, $\mu \sim p$, and much less than the scale of the new physics, $\mu \ll 1/r_e$, then C_0 must be treated to all orders but C_2 gives corrections that are suppressed by pr_e. Higher terms in the potential are similarly suppressed by powers of pr_e and so a systematic power counting does exist in the PDS scheme, although it is not the one suggested by Weinberg.

Moreover the linearly divergent terms are "universal" in the sense that they have the same coefficient in all of the loop integrals, up to powers of p^2. Powers of energy multiplying the integrals and powers of momentum appearing inside the $I_n(E)$ both contribute in the same way, and so there is no distinction between energy and momentum dependence of the potential, as expected for a scheme with a systematic power counting. As noted by Gegelia[21] the same results can also be obtained by keeping the linear divergences and carrying out a momentum subtraction at the unphysical point $p = i\mu$.

PDS as described in Refs.[20] keeps only subtraction terms arising from the linear divergences in the $I_n(E)$. One might ask whether subtraction terms for the higher power-law divergences spoil the power counting. The divergent integral over $x = q^2$ in (19) gives a Γ-function with poles at $D = 3, 1, \ldots, 3 - 2n$ dimensions, corresponding to all the divergences seen with a cut-off regulator in (7). It happens that all except the pole at $D = 3$ are cancelled by zeros of factor $\Gamma(\frac{D-1}{2})^{-1}$ arising from the angular integral. This cancellation would appear to be an artefact of continuing to numbers of spatial dimensions of zero or less, where the angular integration no longer makes sense. If one modifies DR by analytically continuing only the q^2 integral then all the power-law

divergences can be identified and subtracted to leave

$$\frac{I_n(E)}{M} = -\sum_{m=0}^{n} A_m \mu^{2m+1} p^{2(n-m)}. \tag{24}$$

Implemented in this way, PDS leads to a result that looks very like the one obtained above using a cut-off (7), except that the cut-off Λ has been replaced by μ. This is a rather natural result if one regards the scale μ introduced by DR as a resolution scale. The requirement that physics be independent of μ (renormalisation-group invariance) can then be implemented by solving the equations for $C_{0,2}$ in terms of a and r_e, exactly as in Ref.[13,17] for the cut-off case. There is however one important difference between DR (24) and the cut-off (7):[c] the final UV finite, but IR divergent term is absent from the PDS expression. This means that, with $\mu \ll 1/r_e$, one is not forced into having coefficients with the scale dependence (10). Instead the $C_{0,2}$ continue to have the leading scale dependence (23).

In this modified version of PDS, the fitted value of C_0 does shift when C_2 is included in the potential. However this shift is suppressed by μr_e, and so is small for $\mu \ll 1/r_e$. The power counting of Ref.[20] therefore survives the inclusion of higher power-law divergences.

7 Discussion

In the simplified model for s-wave NN scattering by short-range potentials, taking the cut-off to infinity leads to a breakdown[13,17,16] of the power counting proposed by Weinberg[1]. In the absence of a consistent power counting, energy- and momentum-dependent terms in the potential are not equivalent. In particular, as $\Lambda \to \infty$, energy dependence is essential to get the correct effective range.

This is unsurprising if one interprets the (renormalised) short-range potential as simply imposing a boundary condition on the logarithmic derivative of the wave function, as suggested in Manchester some time ago[34]. In the $\Lambda \to \infty$ limit, this boundary condition is imposed at the origin[34,35,36,37,38,39]. Since the logarithmic derivative of the wave function at the origin is just $p \cot \delta(p)$ (cf. the effective range expansion (3)), this implies a zero effective range, unless the boundary condition depends on energy[15].

One alternative is not to take the cut-off to infinity, but to set it to some scale roughly corresponding to the physics that has been omitted from the effective theory, checking that results do not depend too strongly on the precise

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[c]I am grateful to D. Phillips for pointing this out.

value of the cut-off[14]. This form of regularisation is appealing, since we know that the physics described by the contact interactions is not truly zero-range. It has also proved useful in fitting the low-energy behaviour of NN scattering[5,12,19,22]. However in systems with unnaturally large scattering lengths, such an approach is not systematic[17]. It is thus unclear what advantage the parametrisation (4) has over, say, a sum of Yukawa terms[40].

DR with minimal subtraction does lead to a consistent power counting[7]. However in this scheme the divergent loop integrals are all set to zero, and so the K-matrix is equal to the potential. As a result, the range of validity of the momentum expansion is controlled by the scattering length and, in systems with an unnaturally large scattering length, this range is too small to be of practical use[7,11].

This leaves DR with power divergence subtraction[20] as the only hope for a renormalisation scheme that is both useful and systematic. In the PDS scheme, a modified power counting does exist. This is based on taking the renormalisation scale μ to be of the same order as the momenta of interest. The potential and the scattering amplitude both contain terms with all integer powers of the low scale Q (p or μ) starting at order $\mathcal{O}(Q^{-1})$. This is to be contrasted with Weinberg's counting[1], which would be applicable to systems with a natural scattering length. In that, the potential and K-matrix contain only even powers of p starting at order $\mathcal{O}(p^0)^d$.

In the PDS scheme each loop integral contributes one power of p or μ beyond those associated with the vertices in the loop. This is true even if, as suggested above, the scheme is modified to include subtraction terms for all power-law divergences. Thus the power counting of Ref.[20] is not spoiled by keeping higher than linear divergences. One also sees that all iterations of the leading term in the potential are of order $\mathcal{O}(Q^{-1})$, and so should be summed up nonperturbatively. In contrast, higher order terms in the short-range potential can be treated as perturbations (as can pion-exchange if pions are included explicitly in the low-energy theory[20,41]).

8 Renormalisation group ideas

In sections 4 and 5 we have seen that power counting is possible in theories with a natural scattering length, and that the method of regularisation does not affect this conclusion. However, for systems with unnaturally large scattering lengths, cut-off regularisation and DR with PDS lead to quite different results. Although not the one suggested by Weinberg[2], a systematic power counting

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^{d}The T-matrix does contain odd powers of p, but only because of the unitarity term $iMp/4\pi$ in its denominator.

does emerge in the PDS scheme. In contrast no such organisation of the theory is possible when a finite cut-off is used.

If the scale μ introduced by PDS is regarded as a resolution scale, then one can understand why the PDS scattering equations have a rather similar form to those obtained with a cut-off. However, the very different behaviour of the potential in these regularisation schemes is then all the more surprising. As described in sections 4 and 6, the origin of this difference lies in the UV finite pieces of the cut-off loop integrals (7) that are not present in the PDS scheme.[e]

In this section, I discuss low-energy NN scattering from the viewpoint of the renormalisation group (RG) [42,43,44]. These ideas were developed following discussions at the workshop and may eventually help to clarify the differences between the regularisation schemes. In an RG treatment, one demands that the effective theory continue to reproduce physical quantities as the the cut-off scale is lowered and so more and more short-distance physics is "integrated out". For example, one can require that the cut-off theory reproduce the scattering observables for all values of the cut-off by differentiating the expressions for these observables with respect to Λ and setting the derivatives equal to zero. This leads to a set of Wilson-style RG equations for the Λ dependence of the coefficients C_{2n} in the potential. These equations are very similar to the corresponding ones for the μ dependence of the coefficients in the PDS scheme [20]. Alternatively one can avoid writing down differential RG equations by inverting the expressions relating the scattering observables to the C_{2n} and Λ. This approach, which the one followed by the Maryland group in Refs. [13,17], leads directly to the solutions of the RG equations.

Viewed from this perspective, it looks unnatural to try to take the cut-off to infinity as in Refs. [13,17,16], in which case the breakdown of power counting and related problems found in this limit do not rule out the possibility of systematic low-energy effective theory. For example a similar breakdown is expected in mesonic ChPT with a large cut-off (even if the cut-off is imposed in a way that does not violate the symmetries of the theory). Instead one needs to examine the behaviour of the theory as the cut-off is lowered. If the couplings (suitably scaled by powers of Λ) tend to definite values as $\Lambda \to 0$, then this IR fixed point corresponds to a well-defined effective theory.

In the case of scattering by short-range potentials with a natural scattering length, the rescaled couplings $\hat{C}_{0,2}$ defined in (16) tend to the fixed point $\hat{C}_0 = \hat{C}_2 = 0$ as $\Lambda \to 0$. If higher-order terms are included in the potential then one finds that the fixed point is the trivial one, $\hat{C}_{2n} = 0$ for all n. The corresponding effective theory is the one found using either a finite cut-off,

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[e]For more on this, see D. Phillips' contribution to these proceedings [24].

where loop corrections can be treated perturbatively for small Λ, or DR with minimal subtraction, where loop corrections are ignored. As we have seen above, a systematic power counting is possible in a perturbative low-energy theory of this type.

In contrast PDS aims to treat systems with unnaturally large scattering lengths. Strictly speaking, as Λ tends to zero such a theory eventually approaches a fixed point of the type discussed. However, in the region where Λ is much larger that $1/a$ but small compared with $1/r_e$, we can neglect corrections of order $1/\Lambda a$ and consider the approach to a fixed point (if one exists) that corresponds to a theory with infinite scattering length. Such a "quasi-fixed point" is central to the power counting found in the PDS scheme [20]. Let me first consider such systems with a cut-off. In this case, one finds from (15) that to get a finite effective range, both $\hat{C}_0$ and $\hat{C}_2$ must tend to finite values as $\Lambda \to 0$. However there is no power counting: the value of $\hat{C}_0$ at the fixed point changes by a finite amount when $\hat{C}_2$ is included. Similarly both of these coefficients will change if higher-order terms are included in the potential.

The results obtained with a cut-off imply that, if it exists, the fixed point for a system with infinite scattering length is a nonperturbative one; the corresponding values of the $\hat{C}_{2n}$ can only be determined if one includes terms to all orders in the expansion of the potential. The nonperturbative nature of this fixed point may mean that it is not accessible using DR, unlike the perturbative point found for natural systems. Certainly the PDS scheme shows a very different fixed-point behaviour. In this case, all the C_{2n} for $n > 0$ tend to zero and only $\hat{C}_0$ tends to a finite value. As already mentioned, this behaviour is different from that found using a cut-off because DR does not pick up the terms in the loop integrals (7) that are proportional to inverse powers of Λ and so become increasingly important as $\Lambda \to 0$. These terms include the one involving B_1 in (15), which has the effect of driving $\hat{C}_2$ to a finite value in this limit, along with similar ones at higher-orders in the expansion.

By omitting the IR divergent terms from the loop diagrams, the PDS scheme leads to a set of RG equations with a quite different fixed point behaviour from those obtained with a cut-off. It is therefore important to understand the nature of these terms [24]. If they are not artefacts of cut-off regularisation, then the failure to retain these terms in the RG equations suggests that the apparent fixed point of PDS does not really correspond to a well-defined low-energy effective theory. If this is the case, a systematic power counting would only be possible for systems with natural scattering lengths. In the case of interest, s-wave NN scattering, power counting would indeed be powerless.

Acknowledgments

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